ALGEBRA II
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Lesson 0124 - Exponents
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1. EXPONENTS

Let a be a positive real number, and let x be a real number. We ask, what is the meaning of a^x?

1.1. **When x is a positive integer.** Let $n = x$, and assume that n is a positive integer. Then a^n is defined to mean the product of n numbers whose value is a:

$$a^n = \underbrace{a \times \cdots \times a}_{n \text{ times}}.$$

From this, we obtain two significant properties.

(E1) $a^{m+n} = a^m \cdot a^n$
(E2) $(a^m)^n = a^{mn}$

To see this, write

$$a^{m+n} = \underbrace{a \times \cdots \times a}_{m+n \text{ times}} = \underbrace{a \times \cdots \times a}_{m \text{ times}} \times \underbrace{a \times \cdots \times a}_{n \text{ times}} = a^m \times a^n.$$

and

$$(a^m)^n = (\underbrace{a \times \cdots \times a}_{m \text{ times}})^n = \underbrace{(\underbrace{a \times \cdots \times a}_{m \text{ times}}) \times \cdots \times (\underbrace{a \times \cdots \times a}_{m \text{ times}})}_{n \text{ times}} = \underbrace{a \times \cdots \times a}_{mn \text{ times}} = a^{mn}.$$

We wish to extend the meaning of a^x so that it is defined for any real number x, in such a way that the properties **(E1)** and **(E2)** remain true.

1.2. **When $x = 0$.** Consider the case when $x = 0$. We multiply a times a^0; whatever a^0 means, if property **(E1)** is to remain true, we have

$$aa^0 = a^1a^0 = a^{1+0} = a^1 = a.$$

Dividing both sides by a gives

$$a^0 = 1.$$

1.3. **When x is a negative integer.** Consider the case when x is a negative integer, so that $x = -n$ for some positive integer n. For **(E1)** to remain true, we must have

$$a^n a^x = a^{n+x} = a^0 = 1.$$

In this case,

$$a^{-n} = \frac{1}{a^n}.$$

1.4. **When x is rational.** Consider the case when $x = \frac{1}{n}$, where n is a positive integer. For **(E2)** to remain true, we must have

$$(a^{1/n})^n = a^{n/n} = a^1 = a.$$

Thus, $a^{1/n}$ is the unique number whose n^{th} power is a; that is,

$$a^{1/n} = \sqrt[n]{a}.$$

Consider the case when $x = \frac{m}{n}$, where m and n are positive integers. Then **(E2)** produces $a^{m/n} = (a^m)^{1/n}$, so

$$a^{m/n} \underset{1}{=} \sqrt[n]{a^m}.$$

1.5. **When x is irrational.** We now consider the case when x is irrational. This is the hardest step.

Integers are obtained from natural numbers by algebraic considerations (defining subtraction), and rational numbers are obtained from integers by additional algebraic considerations (defining division); however, real numbers are obtained from rationals by geometric considerations (filling in gaps in the number line).

There is an additional property of exponents which is important in this context:

(E3) if $1 < a$ and $r < s$, then $a^r < a^s$

This is true when x is any rational number, and we wish it to remain true for any real number.

We line up all of the rationals by the order relation $<$, and see that there are gaps in the line; so, too, we can line up all of the numbers of the form a^q where q is rational, and see that there are gaps in the line; we hope to fill these gaps by numbers of the form a^x, where x is irrational.

Using calculus, we can show the following.

Fact 1. Let x be an irrational number. Then there exists a unique real number y such that, for every two rational numbers $r, s \in \mathbb{Q}$ with $r < x < s$, then $a^r < y < a^s$.

We define a^x to be the unique y with this property.

1.6. **Exponential Functions.** Let a be a positive real number. Now that a^x is defined for any real number x, we see that, by letting x vary throughout the real numbers, we obtain a function.

The *base a exponential function* is the function a^x. This function has these properties:

(a) $a^0 = 1$
(b) $a^1 = a$
(c) $a^{r+s} = a^r a^s$
(d) $(a^r)^s = a^{rs}$
(e) $r < s \Rightarrow a^r < a^s$, if $a > 1$
(f) $r < s \Rightarrow a^r > a^s$, if $0 < a < 1$

Let $f(x) = x^a$ and $g(x) = a^x$. For f, the variable is in the base, and for g, the variable is in the exponent. We call f and power function, and we call g an exponential function. These functions have very different properties, and the reader is cautioned to keep track of the differences between them.

1.7. **Exercises.**

Problem 1. Use the properties of exponents to simplify the following.

(a) $25^{3/2}$
(b) $343^{-2/3}$
(c) $1024^{2/5}$